Connectivity of fracture systems-a percolation theory approach

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Connectivity of fracture systems—a percolation theory approach

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Abstract. Critical percolation densities have been found numerically for various systems of lines uniformly distributed in the plane. The average number of intersections per line at percolation has also been found and varies only slightly over the cases considered. It may therefore provide a useful rule of thumb for deciding whether a system percolates. An estimate of the critical percolation density from the lattice percolation probability is presented. Possible extensions of the techniques described to three dimensions are discussed.

1. Introduction

In a hard rock, such as granite, water flow occurs almost exclusively through fractures (this term is taken here to include joints, faults, fissures, etc). The pattern of fractures is complicated to such an extent that one cannot expect to model it fully. Some sort of statistical model must be used. One way to approach the problem is to model the whole system (rock and fractures) by an 'equivalent permeable medium' characterised by a permeability tensor. For systems where the scale of the fractures is much less than the scale of interest this is a sensible approach. In granitic rocks this condition does not hold. The distances between major fractures may be metres or tens of metres (Bourke *et al* 1980), with the region of interest tens or hundreds of metres. So, while equivalent permeable medium models are widely used, this possible flaw has been noted (Rae and Robinson 1979). It is argued that because insufficient data on fracture systems exist more complicated models are not justified. Despite this, some work has been done on keeping more of the fractured nature of the physical system in the model (Wang 1980).

In a study of fractured rock three main topics need to be investigated: firstly, connectivity, secondly, flow and thirdly, dispersion. Here we consider the first of these topics. We look at the problem of connectivity in two-dimensional fracture systems. From this we hope to determine the best approach to the full three-dimensional problem. We represent the fractures as line segments of random position with some specified distribution of lengths and orientations. We then ask whether flow paths will exist as the number of fractures per unit area increases.

In § 2 we show the connection with percolation theory, § 3 describes the numerical method used and the computer experiments done, § 4 gives some theoretical predictions of the results and § 5 presents the results. Section 6 presents some conclusions.

2. Percolation theory

In percolation theory one defines a medium to be an infinite set of sites; a fluid flows between these sites along paths which connect certain pairs of sites (these paths are often called bonds). Two types of percolation model are discussed, site percolation and bond percolation. Normally the sites lie on a regular lattice and only bonds between nearest neighbours are considered. Site percolation involves a probability p that any site is open independently of the other sites. A path is then a sequence of connected open sites. In bond percolation the probability is for any bond to be open. A path is a sequence of sites connected by open bonds. In either case a cluster is a set of sites in which any two are connected by a path. It is found that a critical probability p_c exists so that for $p < p_c$ only finite-size clusters exist, but for $p > p_c$ infinite ones appear (Essam 1980).

Our case is in some sense a continuum limit of the site percolation model. Our sites are lines which may be in any position with any length and orientation. Two lines are connected if they intersect. The probability becomes a density (lines per unit area). Because the length scale of the lines is irrelevant, we take (line density) $\times$ (length scale)^2 as the important variable. We call this N, and its critical value N_c.

3. Numerical determination of critical densities

In a numerical model one can obviously not look for infinite clusters. Instead one looks for paths across regions which are taken as large as is practical.

We generate lines of specified length and orientation distributions uniformly in a square. Percolation is said to have occurred when there is a path (cluster) from near one side to near the opposite side. For these evaluations 1000–2000 lines were generated for each run, although for one case 10 runs with around 10 000 lines were done to check that there was no significant effect of the finite size. Lines are generated one by one and a record is kept of all the clusters. Each new line can

(i) form a new cluster (no intersections);

(ii) join an existing cluster;

(iii) unite two or more existing clusters.

As soon as a cluster reaches from side to side the line generation is stopped.

Figures 1–4 show examples of the sort of results obtained. We considered four cases. These were:

(I) Constant line length; angle distributed uniformly in the range $[-\alpha, \alpha]$, for a range of α's.

(II) Constant line length; angle either β or $-\beta$ with probability $\frac{1}{2}$, for a range of β's.

(III) Line length uniformly distributed in the range $[l_{av}(1-f), l_{av}(1+f)]$ for a range of f's; angles uniformly distributed in the range $[-90°, 90°]$.

(IV) Line length uniformly distributed as case III; angles $-45°$ or $45°$ with probability $\frac{1}{2}$.

In each case both N_c and I_c were determined, I_c being the average number of intersections per line at percolation. In each case the length scale used to evaluate N_c was half the average line length.

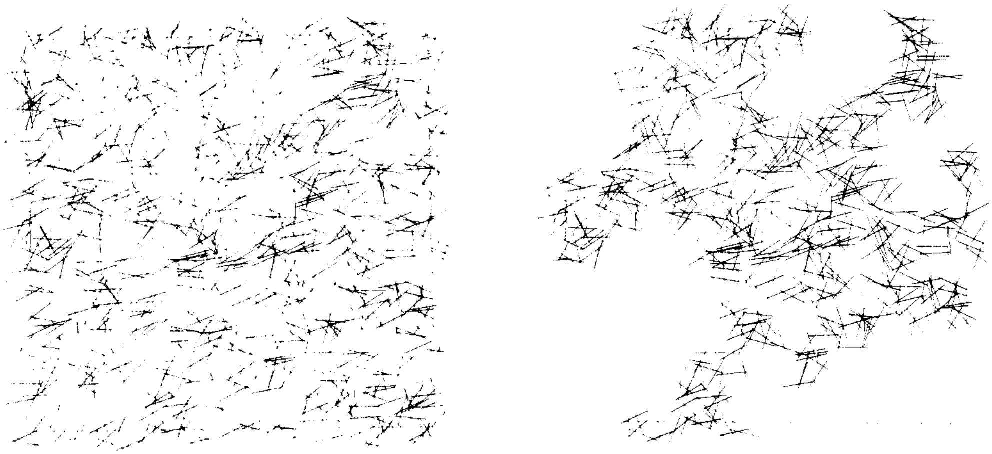


Figure 1. (*a*) Example for constant line length, θ uniform in $(-90°, 90°)$. (*b*) Percolating cluster from constant line length, θ uniform in $(-90°, 90°)$ case.

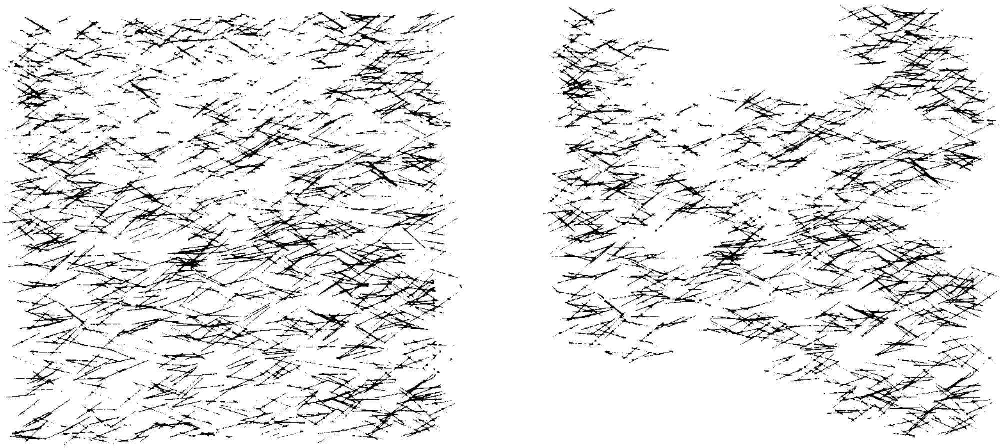


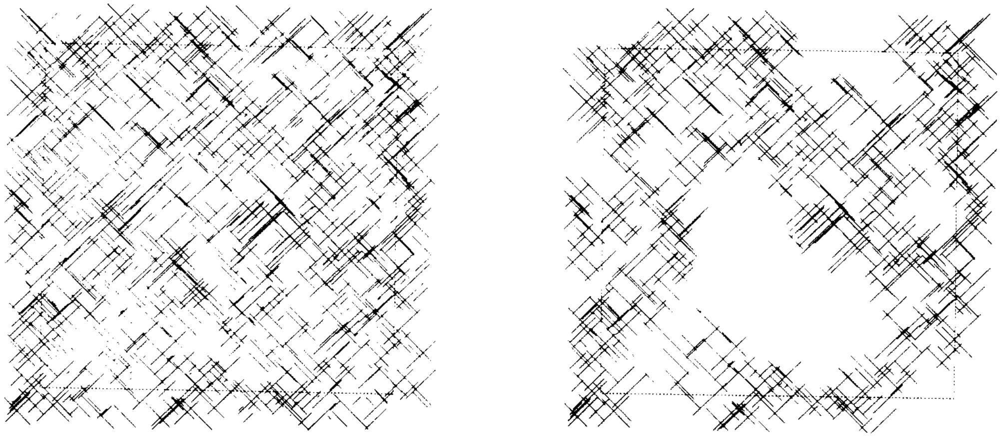
Figure 2. (*a*) Example for constant line length, θ uniform in $(-45°, 45°)$. (*b*) Percolating cluster from constant line length, θ uniform in $(-45°, 45°)$ case.

4. Theoretical predictions

4.1. Relationship between N_c and I_c

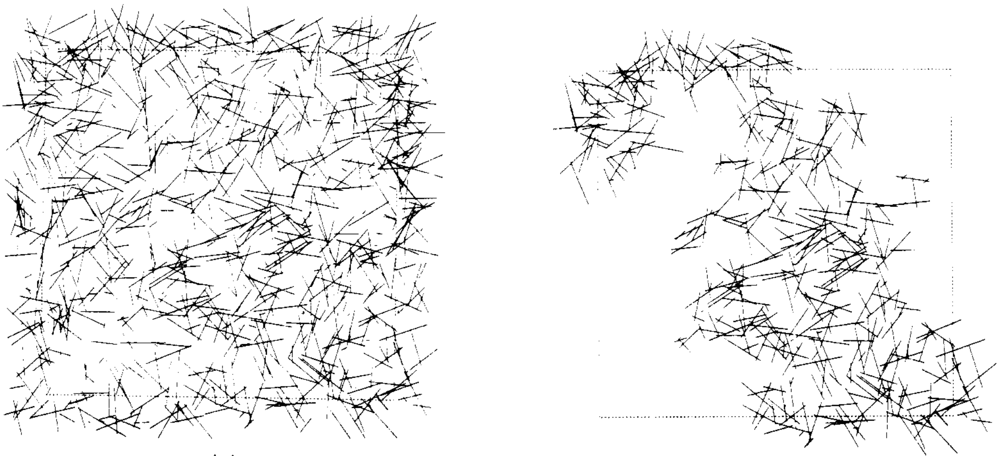
It can easily be shown that in all our cases we expect to find a simple relationship between N_c and I_c:

$$
\begin{aligned}
&\text{(I)} && I_c(\alpha) = N_c(\alpha)(2/\alpha^2)(2\alpha - \sin 2\alpha);\\
&\text{(II)} && I_c(\beta) = N_c(\beta)\, 2 \sin 2\beta;\\
&\text{(III)} && I_c(f) = (8/\pi) N_c(f);\\
&\text{(IV)} && I_c(f) = 2N_c(f).
\end{aligned}
\tag{4.1}
$$



(a) (b)

Figure 3. (*a*) Example for line length uniform in $(0, 2l_{av})$, $\theta = \pm 45°$. (*b*) Percolating cluster from uniform line length, $\theta = \pm 45°$.



(a) (b)

Figure 4. (*a*) Example for line length uniform in $(0, 2l_{av})$, θ uniform in $(-90°, 90°)$. (*b*) Percolating cluster from uniform line length, θ in $(-90°, 90°)$.

4.2. Small-angle limit

We can also show that for cases (I) and (II) we should find

$$\begin{aligned} &\text{(I)} \qquad N_c(\alpha) \sim A/\alpha \qquad \text{for } \alpha \to 0; \\ &\text{(II)} \qquad N_c(\beta) \sim B/\beta \qquad \text{for } \beta \to 0 \text{ where } A \text{ and } B \text{ are constants.} \end{aligned} \tag{4.2}$$

Combining these results with (4.1), we predict

$$\begin{aligned} &\text{(I)} \qquad I_c(\alpha) \to \tfrac{16}{3}A \qquad \text{as } \alpha \to 0; \\ &\text{(II)} \qquad I_c(\beta) \to 4B \qquad \text{as } \beta \to 0. \end{aligned} \tag{4.3}$$

4.3. Distributed line lengths

In cases (III) and (IV) we have a line length l uniformly distributed in the range

$[l_{av}(1-f), l_{av}(1+f)]$ and define N_c by (line density) $\times l_s^2$ where $2l_s = l_{av}$. This gives equal weight to all the lines. The effectiveness of any line depends on how many other lines it intersects and by how much it increases the cluster size. Both these depend on its length. If we take the average effectiveness of our lines and define a constant length line of the same effectiveness l_{eff}, we have

$$l_{eff}^2 = \frac{1}{2f} = \int_{-f}^{f} (1+c)^2 l_{av}{}^2 \mathrm{d}c = (1+\tfrac{1}{3}f^2) l_{av}^2. \tag{4.4}$$

This leads us to predict that $N_c(1+f^2/3)$ is constant, i.e. that

$$\begin{aligned} &\text{(III)} \qquad N_c(f) = N_c(0)/(1+f^2/3); \\ &\text{(IV)} \qquad N_c(f) = N_c(0)/(1+f^2/3). \end{aligned} \tag{4.5}$$

4.4. Estimate of N_c for case II with angles $\pm 45°$

In this case we have orthogonal lines of fixed length l with $N =$ (line density) $\times \frac{1}{4}l^2$. If we cover the plane with squares of side $\frac{1}{2}l$ orientated in the same way as the lines, the average number of lines centred in each square is N. If we now say that two squares are connected if a line centred in one intersects a line centred in the other, we have a regular lattice percolation problem. By postulating that percolation for the regular lattice occurs at the same time as percolation among the lines, we have an estimate for N_c. In the appendix this calculation is done and an estimate $N_c = 1.54$ is obtained.

5. Results

In each case 30 runs were done.

(I) The average N_c and I_c with their standard deviations are presented in figure 5. The predicted N_c for constant $I_c = 3.706$ is also shown, as derived from (4.1) (I).

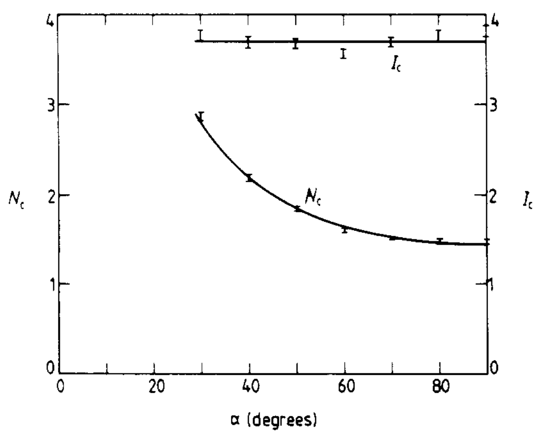


Figure 5. N_c and I_c versus α for case I.

For the $\alpha = 90°$ case we also did 10 runs with around 10 000 lines. These gave $N_c = 1.45$ and $I_c = 3.69$, only slightly less than the results for the smaller runs.

(II) The average N_c and I_c with their standard deviations are presented in figure 6. The predicted N_c for constant $I_c = 3.28$ is also shown, as derived from (4.1) (II).

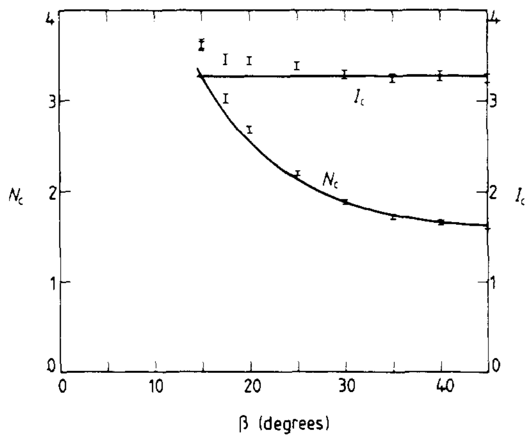


Figure 6. N_c and I_c versus β for case II.

(III) The average N_c and I_c with their standard deviations are presented in figure 7. The predicted N_c and I_c for $N_c(0) = 1.488$ are also shown, as derived from (4.5) (III) and (4.1) (III).

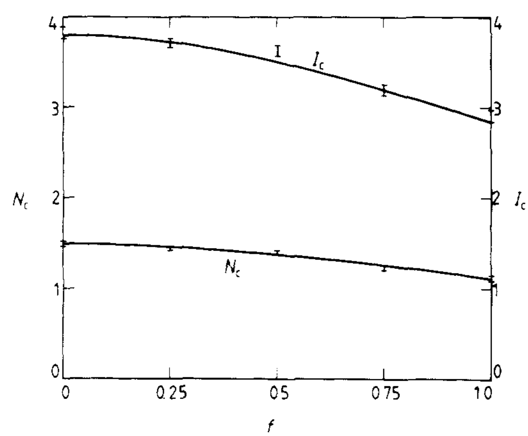


Figure 7. N_c and I_c versus f for case III.

(IV) The average N_c and I_c with their standard deviations are presented in figure 8. The predicted N_c and I_c for $N_c(0) = 1.616$ are also shown, as derived from (4.5) (IV) and (4.1) (IV).

For all cases the part of I_c not affected by the length distribution was in the range 3.2 to 3.8. If this holds, as might be expected, for a wide range of distributions it

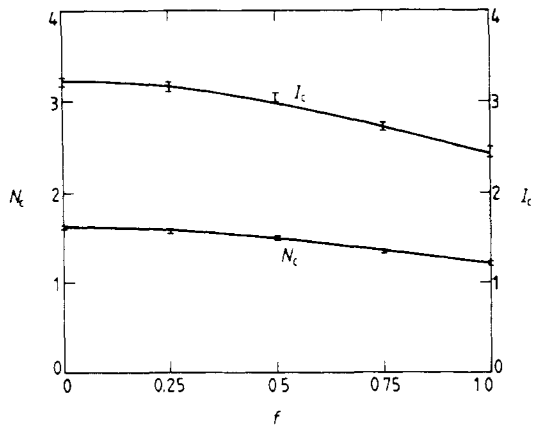


Figure 8. N_c and I_c versus f for case IV.

could provide a useful rule of thumb for estimating critical densities. The estimate of critical density for the orthogonal lines can be extended to three dimensions and a similar rule found.

In a study of percolation for various shapes in the plane, Pike and Seager (1974) give a result for case I with $\alpha = 90$ degrees. They find $N_c = 1.43$, $I_c = 3.63$ when using a slightly different criterion for percolation in the finite region. Given these differences, their result agrees well with ours.

6. Conclusions

The critical density and critical intersection number have been found for a number of angle probability density functions and line length distribution functions. It has been shown that the critical density can be estimated from the known lattice results and that the behaviour for various probability distribution functions for angle and length can readily be predicted.

The critical density and critical intersection number do not vary very much for different statistical properties. This would presumably hold in three dimensions also. In this case an estimate of critical density for three-dimensional systems could be found numerically by looking at just one case. The estimate based on regular lattice results could also be extended to three dimensions.

Acknowledgments

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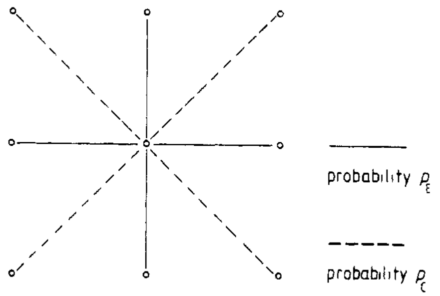
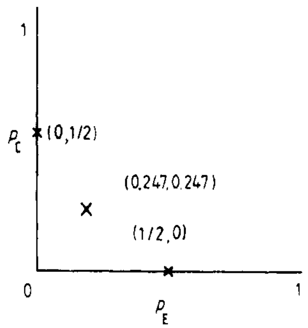
Appendix

(i) Percolation probability for a square lattice with nearest-neighbour bonds probability p_E and next-nearest-neighbour bonds probability p_C:



Percolation occurs when some line in p_C v p_E space is crossed. We know three points of this line, two from the square lattice and the result $p_E = p_C = 0.247$.



From these we can see that

$$p_E + p_C \geq \tfrac{1}{2} \tag{A1}$$

is a good approximation to the condition.

(ii) *Evaluation of p_E and p_C*

Let $P_{rs}^{E/C}$ be the probability that given a square with r lines in one direction is connected to an adjacent one with s lines in the other direction. The E or C superscript denotes an edge or corner adjacency (i.e. nearest- or next-nearest neighbour). Then the probability of connection

$$p_{E/C} = 1 - (1 - Q_{E/C})^2 \tag{A2}$$

where $Q_{E/C}$ is the probability of connection between lines in specified directions,

$$Q_{E/C} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} p_r p_s P_{rs}^{E/C} \tag{A3}$$

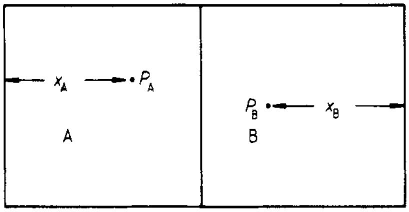
where p_r is the probability that a square has r lines in a given direction.

In fact

$$p_r = \frac{(\frac{1}{2}N)^r \, e^{-N/2}}{r!}. \tag{A4}$$

(iii) *Estimation of P^E_{rs} and Q_E*

Consider unit squares A and B



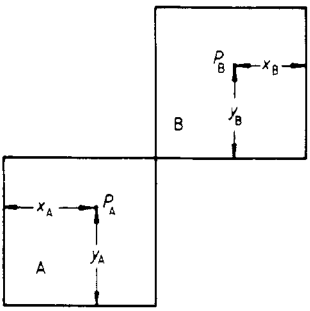
A line centred at P_A in one direction intersects one centred at P_B in the other direction as long as $x_A + x_B \geqslant 1$ (recall that the squares have sides equal to half the line length). With r lines in square A and s in square B we need $\max_{i=1,r}\{x_{Ai}\} + \max_{j=1,s}\{x_{Bj}\} \geqslant 1$. This gives

$$P^E_{rs} = \int_{x_A=0}^{1} dx_A \, r x_A^{r-1} \int_{x_B=1-x_A}^{1} s x_B^{s-1} \, dx_B = \int_{x=0}^{1} dx \, r x^{r-1}(1-(1-x)^s). \tag{A5}$$

Putting this in (A3) with (A4) gives

$$\begin{aligned} Q_E &= \int_0^1 dx \sum_{r=0}^{\infty} r x^{r-1} \frac{(\frac{1}{2}N)^r \, e^{-N/2}}{r!} \sum_{s=0}^{\infty} (1-(1-x)^s) \frac{(\frac{1}{2}N)^s \, e^{-N/2}}{s!} \\ &= \int_0^1 dx \frac{d}{dx}(e^{Nx/2} \, e^{-N/2})(1 - e^{N(1-x)/2} \, e^{-N/2}) \\ &= \int_0^1 dx \, \tfrac{1}{2}N \, e^{N(x-1)/2}(1 - e^{-Nx/2}) \\ &= 1 - (N/2+1) \, e^{-N/2}. \end{aligned} \tag{A6}$$

(iv) *Estimation of $P^C_{r,s}$ and Q_C*



If we take as an approximation the nearest lines in the x coordinate and multiply by $\frac{1}{2}$ for the y coordinate, we get

$$P^C_{rs} = \tfrac{1}{2} P^E_{rs}$$

and

$$Q_C = \tfrac{1}{2} Q_E.$$

(v) *Estimate of* N_c

Now

$$p_E = 2Q_E - Q_E^2,$$
$$p_C = 2Q_C - Q_C^2 = Q_E - \tfrac{1}{4}Q_E^2,$$
$$p_C + p_E = 3Q_E - \tfrac{5}{4}Q_E^2;$$

we want

$$p_C + p_E \geqslant \tfrac{1}{2},$$

i.e.

$$3Q_E - \tfrac{5}{4}Q_E^2 \geqslant \tfrac{1}{2};$$

thus the critical Q_E occurs when $5Q_E^2 - 12Q_E + 2 = 0$; i.e. when $Q_E = 0.180$, this gives

$$1 - (N/2 + 1)e^{-N/2} = 0.180$$

which we can solve to give

$$N_c = 1.54.$$

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